

# ANYONIC LIE ALGEBRAS

DAMTP/97-63

S. Majid<sup>1</sup>

Department of Applied Mathematics & Theoretical Physics  
University of Cambridge, Cambridge CB3 9EW, UK  
[www.damtp.cam.ac.uk/user/majid](http://www.damtp.cam.ac.uk/user/majid)

**Abstract** We introduce anyonic Lie algebras in terms of structure constants. We provide the simplest examples and formulate some open problems.

## 1 Introduction

There have been a number of attempts to generalise supersymmetry in physics to anyonic or fractional statistics. The approach described here originates[1] in the modern theory of quantum groups and braided groups. Although not yet reaching the stage of Lagrangians and field theory, it is mathematically well founded and could perhaps be incorporated into future theories. After a brief introduction to this approach, we describe a natural notion of ‘anyonic Lie algebra’ arising as a special case of [2], but explicitly formulated now through structure constants obeying certain conditions. We also formulate the classification and other problems, and provide some simple examples with  $\mathbb{Z}_3$ -grading.

## 2 Anyspace

The easiest anyonic object to understand is 1-dimensional anyspace. This is the algebra  $\mathbb{C}[\theta]/\theta^n$  with one coordinate  $\theta$  obeying  $\theta^n = 0$  and of degree  $|\theta| = 1$ .  $n = 2$  is a usual Grassmann variable. One may add anyonic variables by

$$\theta'' = \theta + \theta', \quad \theta'\theta = e^{\frac{2\pi i}{n}} \theta\theta' \quad (1)$$

where  $\theta'$  is another copy of the anyonic variable with statistics as shown. One may check that  $\theta''^n = 0$  using properties of  $q$ -binomial coefficients. This additivity provides the key properties analogous to the real line. It has been introduced in [1].

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<sup>1</sup>Royal Society University Research Fellow and Fellow of Pembroke College, Cambridge, England.  
This paper is in final form and no version of it will be published elsewhere.

To be precise one should use the notion of ‘braided groups’[3] or Hopf algebras with braid statistics. Firstly, a braided group is an object in a braided category – a category for which the ‘exchange’ or braiding between any two objects is coherently specified. In our case we use the category of  $n$ -anyonic vector spaces where objects are  $\mathbb{Z}/n$ -graded spaces and the ‘braided transposition’ is

$$\Psi(x \otimes y) = e^{\frac{2\pi i |x||y|}{n}} y \otimes x \quad (2)$$

on elements  $x, y$  of homogeneous degree  $| \cdot |$ . This category has been introduced in [1] as the category of representations of a certain quantum group  $\mathbb{Z}'_n$ . Morphisms preserve degree, although one also (as in SUSY) considers linear maps which are not degree-preserving. Secondly, the group law is expressed as a morphism

$$\Delta : \mathbb{C}[\theta]/\theta^n \rightarrow \mathbb{C}[\theta]/\theta^n \underline{\otimes} \mathbb{C}[\theta]/\theta^n, \quad \Delta\theta = \theta \otimes 1 + 1 \otimes \theta. \quad (3)$$

If we write  $\theta = \theta \otimes 1$  and  $\theta' = 1 \otimes \theta$  (i.e. the generator of the first copy of the anyonic line is called  $\theta$  and the generator of the second copy is called  $\theta'$ ) then we meet the previous notation. We underline  $\underline{\otimes}$  because the two copies do not commute – they have the *braided tensor product algebra*  $(x \otimes y)(w \otimes z) = x\Psi(y \otimes w)z$ . In the formal specification of a braided group we must provide also a counit  $\epsilon(\theta) = 0$  and an antipode  $S(\theta) = -\theta$ .

In general, a braided group means an algebra  $B$  with braiding  $\Psi : B \otimes B \rightarrow B \otimes B$ , a coproduct  $\Delta : B \rightarrow B \otimes B$ , counit  $\epsilon : B \rightarrow \mathbb{C}$  and antipode  $S : B \rightarrow B$ . The easiest way to write the axioms is in a diagrammatic notation[3], see Figure 1, in which we write all maps pointing generally downwards, with  $\cdot = \vee$ ,  $\Delta = \wedge$ ,  $\Psi = \times$ ,  $\Psi^{-1} = \times^{-1}$ . Other morphisms are nodes with the appropriate number of inputs and outputs. In this notation we ‘wire’ the outputs of maps into the inputs of other maps to construct our algebraic operation. Information flows along these wires much as in a computer, except that under and over crossings are nontrivial operators  $\Psi, \Psi^{-1}$ . This is a new kind of ‘braided mathematics’. Note that physicists use such ‘wiring’ notation in the form of Feynman diagrams. In anyonic or braided field theory one would have such diagrams but with under or over crossings being nontrivial operators. The diagrams translate into equations by reading the operations from the top down. Thus an anyonic braided group means a  $\mathbb{Z}/n$ -graded algebra  $B$  and coalgebra defined by  $\epsilon : B \rightarrow \mathbb{C}$  and  $\Delta(x) = x_A \otimes x^A$  say (summation understood over terms labeled by  $A$  of homogeneous degree), coassociative and

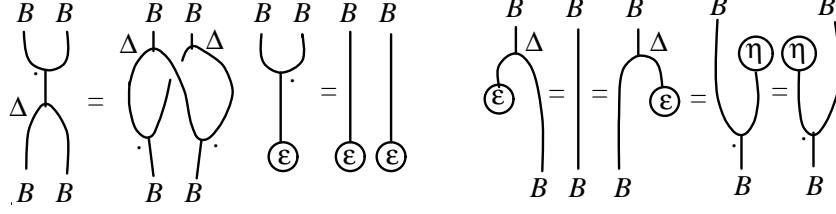


Figure 1: Homomorphism and counity axioms for a braided group

counital in the sense

$$x_{AB} \otimes x_A^B \otimes x^A = x_A \otimes x_A^B \otimes x^{AB}, \quad \epsilon(x_A)x^A = x = x_A\epsilon(x^A) \quad (4)$$

and obeying

$$(xy)_A \otimes (xy)^A = x_A x_B \otimes x^A x^B e^{\frac{2\pi i}{n} |x^A| |x_B|} \quad (5)$$

for all  $x, y \in B$ . The axioms for the antipode are as for usual quantum groups[16] and we omit writing them explicitly.

One has many further structures, such as integration, differentiation

$$\int \theta^m = \begin{cases} 1 & m=n-1 \\ 0 & \text{else} \end{cases}, \quad \partial f(\theta) = \frac{f(\theta) - f(e^{\frac{2\pi i}{n}} \theta)}{(1 - e^{\frac{2\pi i}{n}}) \theta} \quad (6)$$

as well as exponentials, Gaussians[4] and  $\delta$ -functions[5]. In the paper[5] one finds also random walks and Brownian motion on anyspace.

Higher-dimensional anyonic planes can be defined by tensor product. However, if  $B, C$  are braided groups their natural braided tensor product algebra and coalgebra  $B \underline{\otimes} C$  does not form a braided group in the original category. In the diagrammatic notation, one gets ‘tangled up’ if one tries this. Instead, one has to ‘glue’ the categories also[6]. This is known for Hecke-type braid statistics where the braiding has two eigenvalues. In our setting it means we can ‘glue’ anyonic variables of degree either 1 (as for  $\theta$  above) or  $n/2 - 1$  when  $n$  is even. We call the latter type of coordinate  $\eta$ . So a natural higher-dimensional anyspace has an  $s$ -dimensional glued anyonic part  $\{\theta^i\}$  and an  $r$ -dimensional glued ‘fermi-anyonic’ part  $\{\eta^i\}$ . The required ‘glued’ braiding for additivity is, however, no longer just a phase – it involves linear combinations (actually it is the non-standard  $sl_{s|r}$  R-matrix as explained in [6].) Infinite anyonic planes can be found among primary fields in 2-d quantum gravity as explained in [7]. In short, the ‘phase-factor’ anyonic

form (2) is not closed under tensor product and leads one naturally into systems with linear combinations in the statistics!

Although not built by tensor product, there are still plenty of higher-dimensional anyonic braided groups. For example, one has anyonic matrices[8] as a generalization of quantum matrices [9] and supermatrices. Here there are  $N^2$  generators  $\{t_{ij}^i\}$  of degree  $|t_{ij}^i| = f(i) - f(j)$  where  $f$  is a degree in  $\mathbb{Z}/n$  associated with the row or column and

$$e^{\frac{2\pi i}{n}(f(i)f(k)+f(j)f(b))} \underline{R}_{a \ k \ b}^i t_{j \ l}^b = e^{\frac{2\pi i}{n}(f(j)f(l)+f(i)f(b))} t_{b \ l}^k t_{a \ j}^i \underline{R}_{j \ l}^a, \quad \Delta t_{ij}^i = t_{a \ j}^i \otimes t_{a \ k}^a, \quad \epsilon t_{ij}^i = \delta_{ij}^i. \quad (7)$$

We require that  $\underline{R}$  obeys certain anyonic-Yang-Baxter equations[8] (with the result that the anyonic matrices are any-coquasitriangular). One method to obtain  $\underline{R}$  is to start with certain solutions  $R$  of the usual braid or Yang-Baxter equations and ‘transmute’ them. The quotients of such anyonic matrices yield further anyonic braided groups, see [8][10, Appendix]. However, the method probably does not exhaust all anyonic braided groups.

*Problem 1* Classify all low-dimensional anyonic braided groups, i.e find all algebras,  $\Delta, \epsilon$  and optionally antipodes  $S$  for a given  $\mathbb{Z}/n$ -graded vector space  $B$ .

For  $n = 1$  (usual quantum groups) the low-dimensional classification has been done by Radford via computer. There are also some general techniques [11]. For  $n = 2$  (super quantum groups) one has many examples and probably similar techniques.  $n = 3$  would be the simplest truly braided case, with  $\mathbb{Z}/3$ -anyonic braiding and is wide open. Examples are in [1][8].

### 3 Anyonic Lie algebras

The axioms of a braided Lie-algebra are shown in Figure 2 in a diagrammatic form[2][12]. We need an object  $\mathcal{L}$ , a morphism  $[\ , \ ]$ , and (unusually) morphisms  $\Delta : \mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{L}$  and  $\epsilon : \mathcal{L} \rightarrow \mathbb{C}$  forming a coalgebra. While there are one or two recent proposals for a quantum or braided Lie algebra, this is the only one which (a) includes quantum groups and (b) has features of Lie algebras such as tensor product representations, enveloping (braided) bialgebra, etc. We now specialize this theory to the anyonic setting.

Thus, an anyonic Lie algebra means a  $\mathbb{Z}/n$ -graded vector space  $\mathcal{L}$ , say, and degree-preserving maps  $\Delta, \epsilon$  obeying (4) and

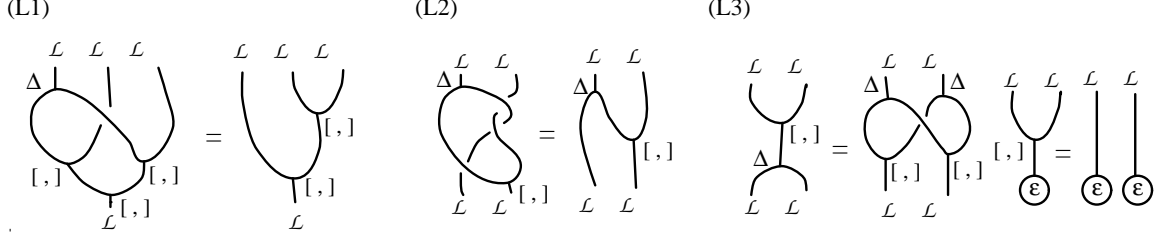


Figure 2: Axioms of a braided-Lie algebra

$$\epsilon([x, y]) = \epsilon(x)\epsilon(y), \quad [x, y]_A \otimes [x, y]^A = [x_A, y_B] \otimes [x^A, y^B] e^{\frac{2\pi i |x^A| |y_B|}{n}} \quad (8)$$

$$x_A \otimes [x^A, y] = x^A \otimes [x_A, y] e^{\frac{2\pi i |x^A| (2|y| + |x_A|)}{n}} \quad (9)$$

$$[x, [y, z]] = [[x_A, y], [x^A, z]] e^{\frac{2\pi i |y| |x^A|}{n}}. \quad (10)$$

This is obtained by reading off the diagrams with the anyonic braiding (2). We write  $\Delta x = x_A \otimes x^A$  as a notation (summation over labels  $A$  understood). The anyonic enveloping algebra  $U(\mathcal{L})$  comes out from the diagrammatic definition in [2] as generated by products of  $\mathcal{L}$  with the relations

$$xy = [x_A, y] x^A e^{\frac{2\pi i |x^A| |y|}{n}} \quad (11)$$

and  $\Delta, \epsilon$  extended as an anyonic braided group by (5).

If we fix a basis  $\{x^\mu\}$  of degree  $|x^\mu| = p(\mu) \in \mathbb{Z}/n$ , then these maps are equivalent to structure constants  $\epsilon, d, c$  defined by

$$\epsilon(x^\mu) = \epsilon^\mu, \quad \Delta x^\mu = d^\mu_{\nu\rho} x^\nu \otimes x^\rho, \quad [x^\mu, x^\nu] = c^{\mu\nu}_\rho x^\rho \quad (12)$$

obeying

$$\epsilon^\mu p(\mu) = 0, \quad d^\mu_{\nu\rho} (p(\mu) - p(\nu) - p(\rho)) = 0, \quad c^{\mu\nu}_\rho (p(\mu) + p(\nu) - p(\rho)) = 0 \quad (13)$$

$$d^\mu_{\alpha\lambda} d^\alpha_{\nu\rho} = d^\mu_{\nu\alpha} d^\alpha_{\rho\lambda}, \quad d^\mu_{\alpha\nu} \epsilon^\alpha = \delta^\mu_\nu = d^\mu_{\nu\alpha} \epsilon^\alpha, \quad c^{\mu\nu}_\alpha \epsilon^\alpha = \epsilon^\mu \epsilon^\nu \quad (14)$$

$$c^{\mu\nu}_\alpha d^\alpha_{\rho\lambda} = e^{\frac{2\pi i}{n} p(\beta)p(\gamma)} d^\mu_{\alpha\beta} d^\nu_{\gamma\delta} c^{\beta\delta}_\lambda c^{\alpha\gamma}_\rho \quad (15)$$

$$d^\mu_{\lambda\alpha} c^{\alpha\nu}_\rho = e^{\frac{2\pi i}{n} p(\lambda)(2p(\nu)+p(\alpha))} d^\mu_{\alpha\lambda} c^{\alpha\nu}_\rho \quad (16)$$

$$c^{\mu\nu}_\alpha c^{\rho\alpha}_\lambda = e^{\frac{2\pi i}{n} p(\beta)p(\mu)} d^\rho_{\alpha\beta} c^{\alpha\mu}_\gamma c^{\beta\nu}_\delta c^{\gamma\delta}_\lambda. \quad (17)$$

The associated anyonic braided group  $U(\mathcal{L})$  is generated by  $1, x^\mu$  with the quadratic relations

$$x^\mu x^\nu = e^{\frac{2\pi i}{n} p(\beta)p(\nu)} d^\mu_{\alpha\beta} c^{\alpha\nu}_\gamma x^\gamma x^\beta \quad (18)$$

A particular ansatz is of the form  $\mathcal{L} = \mathbb{C} \oplus g$ , where  $\mathbb{C}$  is spanned by  $x^0$  with  $p(0) = 0$  and  $g$  is spanned by the remaining  $x^i$ ,  $i \geq 1$ , and

$$\begin{aligned} \epsilon^0 = 1, \quad \epsilon^i = 0, \quad d^0_{00} = 1, \quad d^0_{ij} = d^0_{i0} = d^0_{0i} = d^i_{00} = d^i_{jk} = 0, \quad d^i_{0j} = d^i_{j0} = \delta^i_j \\ c^{00}_0 = 1, \quad c^{00}_i = c^{0i}_0 = c^{i0}_0 = c^{i0}_j = c^{ij}_0 = 0, \quad c^{0i}_j = \delta^i_j. \end{aligned} \quad (19)$$

In this case (13)-(17) for the remaining variables  $p(i), c^{ij}_k$  reduce to

$$1 = e^{\frac{2\pi i}{n} 2p(i)p(j)}, \quad c^{ij}_a c^{ka}_l = c^{ki}_a c^{aj}_l + e^{\frac{2\pi i}{n} p(k)p(i)} c^{kj}_a c^{ia}_l \quad (20)$$

from (16) and (17) respectively. This means effectively that  $n = 1, 2$  and the ansatz recovers the Jacobi axiom for a usual or super-Lie algebra (other  $n$  are equivalent after redefining  $p(i)$ ). The associated anyonic braided group  $U(\mathcal{L})$  is generated by  $1, x^0, x^i$  with the quadratic relations

$$x^0 x^i = x^i x^0, \quad x^i x^j = e^{\frac{2\pi i}{n} p(i)p(j)} x^j x^i + c^{ij}_a x^a x^0 \quad (21)$$

which is the homogenised usual or super enveloping algebra. Setting the central element  $x^0 = 1$  recovers the usual  $U(g)$ . Actually, our objects even for  $n = 1, 2$  are slightly more general than usual or super-Lie algebras because we do not impose any equation like

$$[x^i, x^j] = -e^{\frac{2\pi i}{n} p(i)p(j)} [x^j, x^i]. \quad (22)$$

It turns out that one does not need such antisymmetry for the most important parts of Lie theory or super-Lie theory, although one can impose it additionally at least in the context of the ansatz (19). I do not know a general diagrammatic or axiomatic way to introduce such an antisymmetry condition, however. A related problem: the braided enveloping algebra  $U(\mathcal{L})$  does not usually have an antipode. For this, one needs to quotient it (as in the above ansatz where we set  $x^0 = 1$ ). In all known examples the quotienting procedure is also known, but ‘by hand’.

## 4 Examples of Matrix Type

We now give a family of anyonic Lie algebras  $\mathcal{L}_{N,f}$  going genuinely beyond the super and usual ones. They are a specialization of the general R-matrix construction in [2] to the case

$$R^m_n{}^r_l = \delta^m_n \delta^r_l e^{\frac{2\pi i}{n} f(m)f(r)} \quad (23)$$

where  $m, n, r, l = 1, \dots, N$  and  $f(m) \in \mathbb{Z}/n$  is an arbitrary grading function. We write the  $N^2$ -dimensional vector space  $\mathcal{L}_{N,f}$  in a matrix ('twistor') form where  $x^\mu = x^m_{\dot{m}}$  and  $\mu$  corresponds to the multiindex  $(m, \dot{m})$ . Computing from [2, Prop. 5.2], the induced braiding  $\Psi$  comes out to be the anyonic one with  $p(\mu) = f(m) - f(\dot{m})$ , and

$$d^\mu_{\nu\rho} = \delta^m_n \delta^{\dot{n}}_r \delta^{\dot{r}}_{\dot{m}}, \quad \epsilon^\mu = \delta^m_{\dot{m}}, \quad c^{\mu\nu}_\rho = \delta^m_{\dot{m}} \delta^n_r \delta^{\dot{r}}_{\dot{n}} e^{-\frac{2\pi i}{n} 2f(m)p(\nu)} \quad (24)$$

solves the equations (13)–(17). Equivalently,

$$\Delta x^m_{\dot{m}} = x^m_a \otimes x^a_{\dot{m}}, \quad \epsilon(x^m_{\dot{m}}) = \delta^m_{\dot{m}}, \quad [x^\mu, x^\nu] = \epsilon(x^\mu) x^\nu e^{-\frac{2\pi i}{n} 2f(m)p(\nu)}. \quad (25)$$

The anyonic enveloping algebra  $U(\mathcal{L}_{N,f})$  has the relations

$$x^\mu x^\nu = e^{-\frac{2\pi i}{n} (f(m)+f(\dot{m}))p(\nu)} x^\nu x^\mu \quad (26)$$

and the coproduct extended as an anyonic braided group.

The 1-dimensional case goes naturally with the bosonic choice  $n = 1$  and has the structure

$$[x, x] = x, \quad \Delta x = x \otimes x, \quad \epsilon(x) = 1, \quad |x| = 0, \quad U(\mathcal{L}_{1,f}) = \mathbb{C}[x].$$

It is the structure of the central element  $x^0$  in the ansatz (19) above.

The 4-dimensional case  $\mathcal{L}_{2,f}$  has generators  $x^\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  say. With  $n = 2$  and  $f(1) = 0, f(2) = 1$  as elements of  $\mathbb{Z}/2$ , we have  $a, d$  bosonic and  $b, c$  fermionic, and a super-anyonic Lie algebra

$$\begin{aligned} \epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0, \quad \Delta a = a \otimes a + b \otimes c \\ \Delta b = b \otimes d + a \otimes b, \quad \Delta c = c \otimes a + d \otimes c, \quad \Delta d = d \otimes d + c \otimes b \\ [a, x] = [d, x] = x, \quad [b, x] = [c, x] = 0 \end{aligned}$$

for all  $x = a, b, c, d$ . Note that even although the braiding is the super or  $\mathbb{Z}/2$ -graded one, this  $\mathcal{L}_{2,f}$  is not a usual super-Lie algebra because the coproduct does not have the simple form as in the ansatz in the last section; it is a new kind of super structure. On the other hand, its enveloping super-bialgebra  $U(L_{2,f})$  has relations

$$ax = xa, \quad dx = xd, \quad b^2 = c^2 = 0, \quad bc = -cb$$

for all  $x = a, b, c, d$ , i.e.  $U(L_{2,f}) = M_{1|1}$ , the usual algebra of  $2 \times 2$  super-matrices.

A different choice for  $f$  is with  $n = 3$  and  $f(1) = 0, f(2) = 1$  as elements of  $\mathbb{Z}/3$ , which gives  $\mathcal{L}_{2,f}$  with a similar  $2 \times 2$  matrix of generators but with degrees

$$|a| = |d| = 0, \quad |b| = -1, \quad |c| = 1$$

in  $\mathbb{Z}/3$ . This gives us  $\mathcal{L}_{2,f}$  as an anyonic Lie algebra with  $\mathbb{Z}/3$ -grading. It has the structure

$$\begin{aligned} \epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0, \quad \Delta a &= a \otimes a + b \otimes c \\ \Delta b &= b \otimes d + a \otimes b, \quad \Delta c = c \otimes a + d \otimes c, \quad \Delta d = d \otimes d + c \otimes b \\ [a, x] &= x, \quad [d, x] = e^{\frac{2\pi i}{3}|x|}x, \quad [b, x] = [c, x] = 0 \end{aligned}$$

for all  $x = a, b, c, d$ . The enveloping  $\mathbb{Z}/3$ -anyonic braided group has relations

$$ax = xa, \quad cb = e^{\frac{2\pi i}{3}}bc, \quad b^2 = c^2 = bd = db = dc = cd = 0$$

and coproduct extended to the algebra via (5). One may check that  $ad - cb$  is bosonic, group-like and central and hence may be set equal to 1. Then  $b = c = 0$  and  $a = d^{-1}$ . Hence the quotient of this  $U(\mathcal{L}_{2,f})$  by the determinant relation is the Hopf algebra  $\mathbb{C}[a, a^{-1}]$ .

This second  $\mathcal{L}_{2,f}$  is probably the simplest truly braided anyonic Lie algebra. We can similarly choose various  $n$  and  $f$  for larger matrices, such as  $n = N$  and  $f(i) = i - 1$  in  $\mathbb{Z}/n$  (as for the first  $\mathcal{L}_{2,f}$  above). The  $\mathcal{L}_{3,f}$  of this type has a matrix of generators

$$x^\mu = \begin{pmatrix} a & b_- & b_+ \\ c_+ & d_+ & e_- \\ c_- & e_+ & d_- \end{pmatrix}, \quad |x^\mu| = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

with the matrix coalgebra, and the  $\mathbb{Z}/3$ -anyonic Lie bracket

$$[a, x] = x, \quad [d_\pm, x] = e^{\pm \frac{2\pi i}{3}|x|}x, \quad [b_\pm, x] = [c_\pm, x] = [e_\pm, x] = 0.$$

The enveloping  $\mathbb{Z}/3$ -anyonic bialgebra  $U(\mathcal{L}_{3,f})$  has the following form.  $a$  is central,  $b_\pm^2 = b_\pm b_\mp = 0$  and similarly for  $c_\pm$ , but products of  $b, c$  quasi-commute according to

$$b_\pm c_\pm = e^{\frac{2\pi i}{3}}c_\pm b_\pm, \quad b_\pm c_\mp = e^{-\frac{2\pi i}{3}}c_\mp b_\pm.$$

By contrast, the  $d_\pm$  form a commutative polynomial subalgebra, as do the  $e_\pm$ , but the product of  $d$  with  $e$  variables is zero. The product of  $b$  or  $c$  variables with  $d$  or  $e$  variables is also zero.

*Problem 2* Classify all low-dimensional anyonic Lie algebras by finding all possible  $\Delta, [\ , \ ], \epsilon$  for a given grading  $p(\mu)$  for each basis element of  $\mathcal{L}$ .



*Problem 3* Extend the axioms of a braided-Lie algebra to provide in general for a quotient of  $U(\mathcal{L})$  forming a braided group with antipode.

Actually, one has a matrix braided Lie algebra over all regular points of the ‘Yang-Baxter variety’ as explained in [2] with (23) being special. Deformations of the  $n = 1, 2$  cases are controlled by the classical Yang-Baxter equations and lead to quantum and super-quantum groups. Also, by letting  $|x| \in G$ , an Abelian group and taking braiding  $\Psi(x \otimes y) = y \otimes x\beta(|x|, |y|)$  where  $\beta$  is a bicharacter, one has in a basis  $\{x^\mu\}$  the similar axioms with  $\beta(p(\mu), p(\nu))$  in place of  $e^{\frac{2\pi i}{n} p(\mu)p(\nu)}$  in (13)–(18). This is a general class of braided-Lie algebras which is not exactly anyonic (there need not be a root of unity in the picture) but has a similar ‘phase factor’ form. The ansatz of the form (19) with  $p(x^0)$  the group identity and  $\beta$  skew in the sense  $\beta(g, h) = \beta^{-1}(h, g)$  recovers the axioms of a colour Lie algebra. More generally, we are not limited to skew bicharacters, so we generalize the theory of colour-Lie algebras immediately (anyonic ones being an example of this generalisation). One may similarly use a bicharacter in (23) for matrix braided Lie algebras  $\mathcal{L}_{N,\beta}$ .

More general braided Lie algebras  $g_q$  are associated to the quantum groups [13]  $U_q(g)$  as the appropriate infinitesimal generators. For the  $ABCD$  series we use the R-matrix construction from [2, Prop. 5.2] where  $g_q = \{x^i_j\}$  and  $U(g_q)$  comes out as the braided matrix algebra

$$R_{21}x_1Rx_2 = x_2R_{21}x_1R \quad (27)$$

(in a compact notation) for the appropriate R-matrix (this is [14, eqn. (20)] for the braided matrix algebra  $B(R)$ , written compactly). For the  $A$  series one has  $g_q$  as a deformation of the  $\mathcal{L} = u(1) \oplus g$  ansatz (19). Similarly to that, the corresponding  $U(g_q)$  has to be quotiented by setting a Casimir=1 to obtain (some version of) the Drinfeld-Jimbo quantum enveloping algebras  $U_q(g)$  by this method. To explain this for the general case, we make a ‘quantum-geometry transformation’ [15] and indeed view the  $\{x^i_j\}$  as braided matrix ‘coordinates’ of  $N \times N$  braided matrices  $B(R)$  [14]. Since these were introduced (by the author) as analogues of the quantum matrices  $A(R)$  in [9], they similarly have quotients by (braided)  $q$ -determinant=1 and other relations to braided group coordinate rings  $BG_q$ . The required relations correspond classically to the characterisation of the group  $G$  in  $N \times N$  matrices. Then, via the identification

$$U(g_q) = B(R),$$

we make this same ‘geometrical’ quotienting by braided  $q$ -determinants etc. to get down from  $U(g_q)$  to a version of  $U_q(g)$ . Although it was known from [9] that generators  $l^+Sl^-$  in  $U_q(g)$  obey the relations (27), the prescription to make the required additional quotients when starting from the quadratic algebra (27) is one of the genuinely new results of braided group theory[14][2]. Also, the braided-Lie algebras  $g_q$  are again  $q$ -antisymmetric, although the general axioms for such  $q$ -antisymmetry remain poorly understood. These remarks aim to put the above anyonic Lie algebras into a wider context of braided-Lie algebras and  $q$ -deformation.

In particular, the  $su_2$  R-matrix gives  $su_{2,q}$  as a deformation of  $u(1) \oplus su_2$ , and  $U(su_{2,q}) = BM_q(2)$  is a remarkable identification with  $2 \times 2$ -braided hermitian matrices or  $q$ -Minkowski space in the general ‘twistor’ or spinorial-R-matrix approach of [17] (for the Lorentz algebra) and [14][18][19] (for spacetime algebras based on the relations (27)). The quotienting to  $U_q(su_2)$  then corresponds geometrically to the mass hyperboloid in  $q$ -Minkowski space.

## 5 Anyfields

There are probably many applications of anyonic Lie algebras and anyspaces. One possible application is of course to consider the anyonic variables as ‘organising’ variables for anymultiplets. Thus one can consider fields  $\Phi(x, \theta)$  where  $x$  is a usual spacetime variable. The expansion  $\Phi(x, \theta) = \phi_0(x) + \theta\phi_1(x) + \cdots \theta^{n-1}\phi_{n-1}(x)$  gives the corresponding anymultiplet  $(\phi_0, \dots, \phi_{n-1})$ . Clearly a Lagrangian built from  $\Phi$  could have hidden anyonic properties not visible as a theory of fields  $\phi_i$ . Before writing down such theories one probably needs the answer to the following question:

*Problem 4* What is the anyonic analogue of the super-Poincaré algebra?

Not knowing the answer to this does not, however, stop us from proceeding to more ‘geometrical’ anyfields (without knowing their field theory). For example, anyonic gauge theory is introduced in [20] as an example of braided group gauge theory. There we consider  $n = 3$  and gauge fields  $A(x, \theta)$ . We refer to [20] for details, but the gauge field corresponds to a multiplet of 6 ordinary fields  $(A_1, A_2, a_1, a_2, b_1, b_2)$  where  $A_1$  transforms as a  $U(1)$  gauge field under the first component gauge transformation (itself a multiplet  $(c_1, c_2)$ ) and  $A_2$  in a more complicated

way:

$$A_1 \mapsto A_1 + dc_1, \quad A_2 \mapsto A_2 + dc_2 + (1 + e^{\frac{2\pi i}{3}})(A_1 c_1 - c_1 A_1 - c_1 dc_1). \quad (28)$$

The auxiliary fields  $a_1, a_2, b_1, b_2$  also transform among themselves and  $A_1$ . In fact, there are many variations of such gauge theory according to the choices of differential calculi and gauge group.

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